MIND YOUR p's AND q's! IS PLIMPTON 322 MUSICAL?

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Abstract

In Chapter 9, 'Plato's Musical Trigonometry', of his book *The Pythagorean Plato* (Nicolas Hays, 1978), Ernest McClain proposes a highly imaginative musical interpretation of the cuneiform tablet Plimpton 322. Unfortunately, the author's omission of the first column of the text severely undermines his case. This paper re-assesses the musicality of Plimpton 322 and explores its possible connection with the musical cuneiform tablet CBS 1766.

Reviewing Ernest McClain's book The *Pythagorean Plato*[1] in the *Musical Quarterly* of July 1978, Siegmund Levarie wrote:

> There is something very special about this book as regards both method and contents. Scholars in various fields – mathematics, philology, political science, education, philosophy, music – can profit from it significantly and in due time are likely to recognise it as essential to further thought in their respective disciplines.

Many of us will have benefitted enormously in our research from Ernest's insights in this book and also in *The Myth of Invariance* [2]. But since 1978, cuneiform scholars, historians of mathematics and musicologists, in particular, have made considerable further progress in the proper understanding of various musical and mathematical cuneiform texts. This is especially true in the case of Plimpton 322.

Plimpton 322

The clay tablet with the catalogue number 322 in the G. A. Plimpton Collection at Columbia University is probably the best known, and also the most notorious of all mathematical cuneiform tablets.

So much so, that the mathematician, Eleanor Robson[3], considered it necessary to issue a 'warning' to anyone attempting to interpret the tablet outside its appropriate historical context. In Chapter 9 of *The Pythagorean Plato*, a chapter headed 'Plato's Musical Trigonometry'; Ernest McClain proposes a highly imaginative musical interpretation of Plimpton 322. He views the tablet as a 'prototype' for what he calls 'Plato's musical and trigonometric constructions'. Unfortunately, in doing so, the author has omitted the first column of the text, which is crucial to any proper understanding of this mathematical tablet. Thereby, McClain has undermined his thesis by means of an 'own goal'!

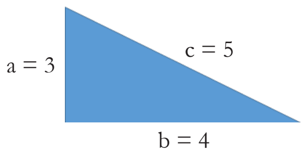
Perhaps the most sober commentary on Plimpton 322, written by someone with a deep understanding of the rest of the Old Babylonian mathematical corpus, can be found in Appendix 8 of Joran Friberg's *A Remarkable Collection of Babylonian Mathematical Texts*[4]. Friberg interprets the text as 'a table of parameters for **igi-igi.bi** problems' - problems involving a reciprocal pair of sexagesimal numbers (**igi** and **igi.bi**) such that their product is equal to 1 or any power of 60. The tablet therefore describes a series of shortcuts[5] for calculating the sides of squares, their diagonals, and hence the sides of Pythagorean triangles also. A number of Seleucid mathematical texts include problems for the solution of which the 'Pythagorean rule' for triples is required. Historians of mathematics tend therefore to assume that the 'Pythagorean rule' was known to Seleucid mathematicians, and probably earlier. Hoyrup[6] suggests the terminology 'Pythagorean rule', rather than 'Pythagorean Theorem', since, strictly speaking, the process is neither Pythagorean, nor a theorem. While accepting all these conclusions, Goncalves[7] adds that there are also some Seleucid mathematical problems where Pythagorean triples have to be obtained without using the 'Pythagorean rule', but rather by using a standard Mesopotamian method for dealing with pairs of reciprocal numbers.

But to go back to basics, we all remember from our schools days that Pythagorean triangles are right-angled triangles in which the square on one side, plus the square on the other, equals the square on the hypotenuse. Figure 1 below, reminds us of this, and all about the p and q values related to Pythagorean triples[8]:

p and q are pairs of integers, which generate Pythagorean triangles where:

$a = p^2 - q^2$; $b = 2pq$; $c = p^2 + q^2$

When p/q = 2/1 (the ration of the musical octave) the triangle is 3:4:5



$a = 2^2 - 1^2 = 3$
$b = 2 \times 2 \times 1 = 4$
$c = 2^2 + 1^2 = 5$

$3^2 + 4^2 = 5^2$ (9 + 16 = 25)

Figure 1.

Plimpton 322 comprises fifteen rows of four extant columns of data related to the 'Pythagorean rule'. Its format, a landscape-orientated headed table, was a formal feature of administrative tables from Larsa during the period 1790-1780 BC[9]. The first column of the tablet is damaged, so that mathematicians cannot be sure whether all the numbers in it should start with 1. Above the four columns are two lines of text. For convenience, the numerical lines of the tablet (3-17) have been re-numbered 1-15 in figure 2. This figure lists the 'p' and 'q' values for each of the Pythagorean triangles associated with the text[10].

Line	p	q	
1	12	5	Musical ratios and acoustical commas
2	64	27	
3	75	32	
4	125	54	
5	9	4	Formula: $(p^x/q^x)x(q^y/p^y)$
6	20	9	e.g. lines 1-8: $(p^1/q^1)x(q^8/p^8)$=
7	54	25	(12/5)x(15/32)=9/8, the Major tone
8	32	15	
9	25	12	
10	81	40	
11	2	1	
12	48	25	
13	15	8	
14	50	27	
15	9	5	

Figure 2

One of my recent discoveries has been that by applying the formula shown on the right in figure 2 to pairs of p's and q's, one can generate all the ratios required to define musical pitches in both Pythagorean and Just tuning, within a perfect fourth, together with the appropriate acoustical commas. The formula consists of two p and q pairs, the first of which is divided by the second, using the Babylonian method of multiplying the ratio of the first pair by the reciprocal of the second pair. This is demonstrated in figure 3 (for intervals) and figure 4 (for 'commas'). To play my own 'devil's advocate', however, this coincidence may be no more than the outcome of exclusively using sexagesimal numbers –that is, numbers in the form $2^p3^q5^r$, which seem to have some relationship to the pure-tone tunings of the natural acoustical harmonic system[11], while harmonics with non-sexagesimal numbers in the series produce more ambiguous tunings.

Major tone	9:8	1-8	12/5x15/32=9/8
Minor tone	10:9	11-15	2/1x5/9= 10/9
Pythagorean leimma	256:243	2-5	64/27x4/9=256/243
Just diatonic semitone	16:15	1-5	12/5x4/9 = 16/15
Just chromatic semitone	25:24	11-12	2/1 x25/48 = 25/24
Just major third	5:4	5-15	9/4x5/9= 5/4
Just minor third	6:5	1-11	12/5x1/2=6/5

Figure 3

Pythagorean tuning standardised the ratios of the tone at 9/8, and the semitone at 256/243, respectively. In Babylonian Just tuning there were two kinds of tones: major 9/8 and minor 10/9. Just tuning also called for three kinds of semitone: 16/15; 25/24; and 27/25. The last two semitones mentioned, taken together (25/24 x 27/25), add up to a major tone (9/8); while the 25/24 semitone, together with a 16/15 semitone, make up a minor tone (10/9).

Figure 4 shows how the acoustical commas, which create problems for both Pythagorean and Just tuning, can also be derived from the p and q values related to various lines of Plimpton 322. The one exception is the Pythagorean comma, the difference between seven octaves and twelve perfect fifths ($3^{12}/2^{19}$ = 531441/524288).

Comma	Ratio	Lines	Calculation
Syntonic comma	81:80	1-2	12/5x27/64=81/80
(Also lines 3-4, 5-6, 7-8, 10-11,13-14)			
Diesis	128:125	8-9	32/15x12/25=128/125
Diachisma	2048:2025	2-3	64/27x32/75=2048/2025
Lines 4-5, 6-7, 9-10, 14-15 all generate 250/243 Quartertone? A third form of a just semitone is: 27:25 27/25x25/24=9/8; 25/24x16/15=10/9; 250/243x82/80=25/24			

Figure 4

Key to the 'commas'

The Syntonic comma is the difference between a pure major third (5:4) and a Pythagorean ditonic third $(9/8)^2$.

A Diesis is the modern acousticians' term for the difference between three pure major thirds $(5/4)^3$ and an octave (2:1).

A Diachisma (2048:2025) defines the discrepancy between two enharmonic pitches – e.g. C natural and B sharp – when, for instance, two pure thirds (5:4), i.e. G sharp and B sharp, are added to four perfect fifths above C (G, D, A, E).

CBS1766

CBS 1766 originally photographed by Hilprecht[12] and published in 1903 with the description 'an astronomical tablet from the Temple Library', was saved from oblivion and republished with a mathematical interpretation by Horowitz[13] in 2006. A year later, Waezeggers and Siebes[14], noticing that around the seven-pointed star on this tablet, there stand the names of seven of the nine musical strings listed in both Sumerian and Akkadian in UET VII 126, proposed an alternative musical interpretation of the tablet as a visual aid for the tuning of seven heptachords on a seven-stringed instrument. The names of the seven strings also reappear in CBS 10996. Furthermore, the present author[15] has subsequently published a detailed musicological interpretation of the seven-pointed star on CBS 176. However, the date and provenance of the tablet CBS 1766 still remain uncertain. Joran Friberg[16] has pointed out that its format is similar to that of Plimpton 322. While Robson[17] has commented that the data on Plimpton 322 'is laid out in a landscape-orientated table with a final heading MU.BI.IM' ('its name') for the non-numerical data'. She adds importantly that these were 'formal features of administrative tables from Larsa during the period of rigorous standardisation in 1790-80s BCE'. In the light of all this, Friberg concludes that 'CBS 1766, like Plimpton 322, in all probability is an Old Babylonian text from Larsa, dating to the period 1790-1780 BC'. He also acknowledges that CBS 1766 is 'a text with a mixed topic' – that is, mathematical, describing a method for drawing heptagons; and musical, for describing a system for the musical tuning of the Mesopotamian heptachords. This, of course, raises a further question: could Plimpton 322, like CBS 1766, be a 'mixed topic' document, dealing not only with squares and Pythagorean triangles, but also with ratios for musical tuning?

I consulted two mathematicians over this question: Joran Friberg and Carlos Goncalves. Both confirmed that, in their opinion, Plimpton 322 is an exclusively mathematical text. However, in addition to the musical significance of the p and q values for the Pythagorean triangles associated with Plimpton 322, which have already been demonstrated in figures 3 and 4, there is another musically curious co-incidence. The p/q values of lines 1 and 15 of Plimpton 322, that is 12/5 x 5/9 = 4/3, span the musical interval of a perfect fourth. If we assign 3600 = 60^2 =1 as the tone-number for line 1 of the text, and then calculate the tone-numbers for the remaining lines, afterwards, adding the tone-numbers of the reciprocal of the Plimpton perfect fourth, thereby extending the original fourth to the length of a heptachord, the tone-numbers of lines 15, 11, 8, 1, 7, 12, 15 become 2700, 3000, 3200, 3600, 4000, 4500, 4800. Interpreted as increasing string-lengths, such a series of tone-numbers would sound the falling heptachord embubum. This is shown in figure 5, where the right-hand column represents the initial perfect fourth of Plimpton 322, and the left-hand column the added reciprocal perfect fourth. The two conjunct tetrachords together form the heptachord.

If each of the tone-numbers in figure 5 is then divided by 100, they match the numbers taken from the Mesopotamian standard tables of reciprocals, dating from the same period as Plimpton 322 and CBS 1766, which define the pitches of the heptachord embubum in my musical and mathematical interpretation of the seven-pointed star on CBS 1766. See figure 6.

line	Pitch	Tone number			
15	D	2700	1	A	3600
14			2		
13			3		
12			4		
11	C	3000	5		
10			6		
9			7	G	4000
8	B	3200	8		
7			9		
6			10		
5			11		
4			12	F	4500
3			13		
2			14		
1	A	3600	15	E	4800

To line 1 assign tone number 3600 = 60^2 = 1, then:
Lines 15, 11, 8, 1 (left) and 1, 7, 12, 15 (right)
embūbum

Figure 5

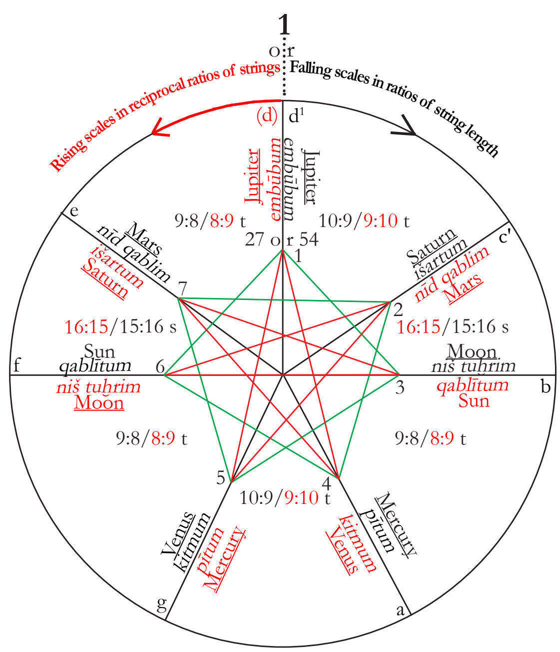


Table 7

CBS 1766 as a Tone Circle and Planets

Notes and Key:
The tritonic tuning procedure of UET 7, 74 can be applied to the falling scales:
Dichords in CBS 10996 — t = tone
Initial tuning (fifths and fourths) — s = semitone
Fine tuning (thirds and sixths) — c' = middle c
Figures in red indicate reciprocals (inverse) scales

Figure 6 is a version of table VII from an earlier paper, updated by the addition of the names of the seven Chaldean planets. The planets are not directly relevant to the current argument, but they do provide me with an excuse to quote from the abstract of an article by Francesca Rochberg , which contains an important insight about Babylonian astronomy and its place in the history of science.

> 'This paper traces the reception of Babylonian astronomy into the history of science beginning in the early mid- twentieth century when cuneiform astronomical sources became available to the scholarly public. The dominant positivism in philosophy of science of this time influenced criteria employed in defining and demarcating science by historians, resulting in a persistently negative assessment of the nature of knowledge evidenced in cuneiform sources'.

The same could be said about music in cuneiform sources. For the reductionism of science has led to music being viewed by most cuneiform scholars merely as a social factor in Mesopotamian society.

The ancient science of harmonics has been largely ignored. It has been left to the insight of daring thinkers like Ernest McClean to grasp how musical ratios in ancient times may have served as a paradigm for a variety of forms of thought across the whole of the Near East[19]. Since the ancient Greeks projected the musical ratios which defined their scales onto the heavens, as an explanation of 'the music of the spheres', so we must move forward over a thousand years from the time of Plimpton 322 to find evidence supportive of the idea that music underpinned the thinking of many ancient cultures. For the third of my co-incidences, therefore, I turn to Plato's *Republic*, where he writes about 'two harmonies'.

Plato's 'Two Harmonies'

In Books VIII and IX of the *Republic*, Plato sketches an outline for an ideal political science. He lists five kinds of regime and explores the characteristics displayed by typical leaders of each of them, from a 'philosopher-king' to a 'dictator'. According to Plato, regimes deteriorate as the leaders born into them grow increasingly out of tune with some musical and mathematical paradigm laid up in heaven. It is at 545d that Socrates decides to call upon the Muses to explain how change and conflict arise in a community. The Muses' oracular answer then introduces the idea of the 'two harmonies'. Societies detiorate as the timing of their leaders' births deviates in some way from these two harmonies. Figure 7 quotes part of the relevant passage (*Republic* 546 b-c) in the translation by Allan Bloom[20]. Below the text, stand: (1) James Adam's explanation of the riddle in the Muses' reply; and (2) my calculation of the two harmonies from arithmetic evidence in the previous sentence.

> '...the root four-three mated with the five, thrice increased, produces two harmonies. One of them is equal an equal number of times, taken one hundred times over. The other is of equal length in one way but is oblong; on one side, of one hundred diameters of five, lacking one for each; on the other side, of one hundred cubes of three, This whole geometrical number is sovereign of better and worse begettings.
>
> Adam:
>
> (1) (7^2 x 100) - (1 x 100) = 4900 - 100 = 4800 =3600 x 4/3
>
> (2) $(\sqrt{50})^2$ x 100 - (2 x 100) = 5000 - 200 = 4800 = 3600 x 4/3
>
> (3) 3^3 x 100 = 2700
>
> Crickmore:
>
> (3/4:1::1:4/3)x6=(9:12::12:16)x5=45:60::60:80) x 60 = 2700 : 3600 :: 3600 : 4800.
>
> 2700 x 4800 = 3600^2 = 12,960,00 = 60^4

Figure 7

James Adam[22] describes this passage as 'notoriously the most difficult in his writings'. Modern classical scholars seem to have found it so difficult that, Robin Waterfield[23] in his 1993 translation of the *Republic*, finds it necessary to comment: 'Scholars nowadays largely ignore the passage'!

'*The root four-three*': this phrase refers to the superparticular ratio 4:3 and its reciprocal 3:4. Setting these ratios either side of unity, functioning as a geometrical mean, produces: 3/4 : 1 :: 1 : 4/3. Multiplied by 12, to convert its terms to integers, gives us 9 : 12 :: 12 : 16, which is the underlying arithmetic of the Mesopotamian heptachords: namely, two conjunct tetrachords.

'Mated with the five, thrice increased':

This instruction indicates that the original number pattern has to be multiplied in turn by 12, (to express it in integers) and then by 5, and 60.

'*Produces two harmonies*':

The two harmonies resulting from this multiplication are therefore 2700 : 3600 and 3600 : 4800. 3600 is the geometrical mean ($\sqrt{ab}$) between 2700 and 4800. In his *Timaeus* 31c, Plato calls the geometrical mean the 'fairest of bonds'. 36002 = 12,960,000. It is a square number, and also Plato's *'sovereign geometrical number of better and worse begettings'.*

2700 x 4800, geometrically an oblong, also produces 12,960,000 (60^4).

We are now in a position to solve Plato's riddle. '*Equal an equal number of times, taken one hundred times over*' refers to 62 x 100 = 3600. One side of the oblong is to be '*one hundred rational diameters of five, lacking one for each*', that is, (72 x 100) – (1 x 100) = 4900 -100 = 4800. The '*rational diameter*' of 5 refers to the nearest rational number to the real diameter of a square whose side is 5. The nearest rational number to $\sqrt{50}$ is 7, the square-root of 49. Thus (72 x 100) – (1 x 100) = 4900 – 100 = 4800. The alternative calculation, using '*irrational diameters*', is shown in item (2) of Adam's explanation in figure 7. The concept of rational number lies at the heart of the arithmetic of ancient music and cosmology. And they had no zero. *Finally, 'one hundred cubes of three' are equal to 2700.*

We have already met these 'two harmonies', 2700: 3600 and 3600:4800, among the tone-numbers defining the Mesopotamian heptachord embubum in Figure 5, and derived from Plimpton 322; and again, divided by 100, in figure 6, derived from CBS 1766. It seems reasonable to assume that Plato's audience would have understood this passage, and fully enjoyed the wit of the Muses' riddle, just as we might enjoy a clever crossword clue. If this be so, the four numbers 2700, 3600, 4800 and 12,960,000 are likely to have been part of a long-standing geometrical and musical tradition.

Plato believed that virtue and happiness in an individual, or a regime, depended on its conformity with a paradigm or divine pattern laid up in heaven (Republic, 472c). For this archetypal idea, Plato probably had in mind the 'musical tetractys'. The 'musical tetractys' expresses the arithmetic underlying the seven Greek octave species: namely, two disjunct tetrachords with a tone (9:8) in-between. It arises from treating the octave (2:1) as 'two extremes', and placing the arithmetic and harmonic means between them. Thus a : (a + b)/2 :: 2ab/(a+b) : b; or, numerically: 2 : 3/2 :: 4/3 : 1. Converting this to integers through multiplication by 6, gives us the 'musical tetractys' in its standard form: 6 : 8 :: 9 : 12. Instrumental strings of these lengths would sound the pitches of the tonic, dominant below, subdominant below and tonic octave below in any chosen key. As we have noted earlier, the Mesopotamian equivalent, corresponding to their seven heptachords, and also to the 'two harmonies' of the Muses, would be two conjunct tetrachords. I suspect, therefore, that if cuneiform scholars, Greek scholars and historians of science would only bear these proportional patterns in mind when studying ancient sources, they might more quickly reach a deeper understanding of the nature of early scientific thought in the ancient cultures of the Near East.

What then can we conclude from all this analysis and speculation?

Conclusions

1. The cuneiform tablet CBS 1766 explicitly mentions the names of seven of the nine strings which are also listed in UET VII 126. We may therefore confidently conclude that CBS 1766 is a musical as well as a mathematical text.

2. Since there are no such explicit evidential references to music in Plimpton 322, it is prudent to conclude that its musical aspects are 'accidental', that is coincidences arising from the use of sexagesimal numbers.

3. Nevertheless, the ratios required by the ancient science of harmonics to define musical pitches, both in Pythagorean and Just tuning, can be derived from the p and q values of fifteen of the possible triangles, generated from all possible values of p and q less or equal to 125, with q > p . Some historians of mathematics have associated these triangles (though possibly anachronistically) with Plimpton 322.

4. Finally, as Ernest McClain has so amply demonstrated, the inherent musicality of Mesopotamian sexagesimal numbers has generated a host of harmonic resonances throughout the writings of Plato.

Notes and references

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5 Sq (igi + igi.bi/2) = sq c, where c is the diagonal, and if 1 is subtracted from the square (igi + igi.b/2) then sq (igi.igi.bi)/2 = sq a. Thus the text is also dealing with right-angled triangles where c = (igi + igi.bi)/2; b = 1; a = (igi.igi.bi)/2, and c + a = igi; c-a = igi.bi; sq c –sq a = igi.igibi = 1 (60).

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10 Of the fifteen p/q ratios related to Plimpton 322, only one (81/40) does not appear in McClain's list of 'Symmetrical Trigonometric Functions within 430,000' (The Pythagorean Plato, pp. 120-1, Figure 41) which he speculatively identifies with the 37 Guardians of Plato's Magnesia (Laws, 752e). However, McClain's list also includes two further p/q ratios which are not related to the data contained in Plimpton 322: 125/64 and 288/125. Of all the possible regular values of p and q, with p less than or equal to 125 (and q< p), only one (125/64) is unrelated to the numbers in Plimpton 322.

11 For instance, the pitches of a natural trumpet in c numbered 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, which are all sexagesimal numbers, produce pure tones, whereas those numbered 7, 11, 13 and 14 which are not sexagesimal, produce more ambiguous pitches.

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